The Equivalence of Mass and Energy: Blackbody Radiation in Uniform Translational Motion

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Uniform translational motion causes a hollow cylinder filled with blackbody radiation to gain an apparent inertial mass. Based on an electrodynamic analysis, Hasenöhrl determined that the relationship between the velocity-dependent apparent inertial mass (m) and the energy (E) of radiation was given by $E = \frac{3}{4}mc^2$. The relationship between mass and energy derived dynamically by Hasenöhrl differs from $E = mc^2$ derived kinematically by Einstein. Here I use the relationship between the energy and linear momentum (p) of a photon ($E = pc$) and the dynamic effects produced in Euclidean space and Newtonian time by the Doppler effect expanded to second order to show that Einstein's equation for the mass-energy relation is the correct one for a radiation-filled cylinder in uniform motion. The increase in linear momentum caused by the velocity-induced Doppler effect expanded to the second order results in an increase in the internal energy density of the radiation. The velocity-dependent increase in the internal energy results from both an increase in temperature and an increase of entropy consistent with Planck's blackbody radiation law. The velocity-induced increase in the internal energy density is equivalent to a velocity-dependent increase in the apparent inertial mass of the radiation within the cylinder.

1. Introduction

As a result of a kinematical analysis, Einstein [1,2,3] derived the following equation to describe the relationship between the energy (E) and inertia (m) of light:

$$E = mc^2 \tag{1}$$

By contrast, Hasenöhrl [4,5,6] took a dynamic approach to determine the relationship between the energy and inertia of light, by considering the work done by the anisotropic radiation pressure [7] inside a uniformly translating cylinder (Fig. 1). Hasenöhl's equation describing the relationship between the energy and inertia of blackbody radiation in a uniformly moving cylinder differs from Eqn. (1) by an algebraic prefactor:

$$E = \frac{3}{4}mc^2 \tag{2}$$

The goal of this paper is to show that Eqn. (1) is the correct equation for describing the relationship between the energy and inertia of radiation for a uniformly translating blackbody radiation-filled cylinder although the assumption of relative time used by Einstein to derive Eqn. (1) is unnecessary.

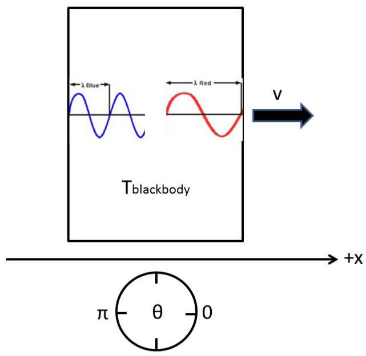

Fig.1: Hasenöhrl's *gedankenexperiment*. A cylinder with matte black and adiabatic walls and constant and invariant volume containing blackbody radiation characterized by absolute temperature T moves through a vacuum at absolute zero. According to the Third Law of Thermodynamics, absolute zero is unattainable, so in any real experiment, the cylinder will have to move through a vacuum with a finite temperature, which itself will result in an optomechanical friction due to the Doppler effect expanded to the second order and the need for additional applied potential energy to move the cylinder at a given uniform velocity for a given time. The orientation of θ relative to the *x axis* is shown, the orientation of φ is perpendicular to the *x axis* and parallel to the end walls. The length (l) of the cylinder is equal to twice the radius (r) of an end wall so that $2\pi rl = 2\pi r^2$. With this geometry, one-half of the photons interacting with a wall interact with an end wall. The time it takes a photon to propagate from end wall to end wall is l/c.

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2. Results and Discussion

The derivation I give here is based on the postulates of Euclidean space, Newtonian time, and the Doppler effect expanded to second order [8]. I assume that the walls of the cylinder are adiabatic, that the volume is constant and invariant, that the thermodynamic system is closed, and that energy and linear momentum are conserved. I discuss a situation where the walls are matte black and are perfect absorbers and emitters. I consider the possibility that the portion of the potential energy applied to the moving cylinder that is not transformed into kinetic energy is transformed at the moving matte-black walls into a thermodynamic potential energy such as the internal energy (U).

According to the work-energy theorem, the applied potential energy ($PE_{applied}$) needed to move an object quasi-statically from a state of rest to another state with higher uniform velocity ($\boldsymbol{v}$) for time interval (dt) is given by the following formula:

$$PE_{applied} = \int_{t_1}^{t_2} \boldsymbol{F_{applied}} \boldsymbol{v}\, dt \tag{3}$$

Where, $\boldsymbol{F_{applied}}$ is the average force applied over time interval dt.

If one were to move a blackbody radiation-containing cylinder at constant velocity through a vacuum at absolute zero where there is no external material - nor radiation-friction, the radiation inside the cylinder would still add a velocity-dependent apparent inertial mass to the cylinder. This is because the radiation inside the cylinder that strikes the back wall would be blue-shifted while the blackbody radiation inside the cylinder that strikes the front wall would be red-shifted (Fig. 1). The Doppler-induced frequency shift expanded to second order of each photon that makes up the blackbody radiation due to relative motion is given by the following equation [8]:

$$\nu_v = \nu_o \left[\frac{1 - \frac{v \cos\theta}{c}}{\sqrt{1 - \frac{(v\cos\theta)^2}{c^2}}} \right] \tag{4}$$

Where, ν_o is the frequency of the photon at the peak of Planck's blackbody radiation curve that describes the blackbody radiation in the cavity at rest ($v = o$), ν_v is the frequency of the photon at the peak of Planck's blackbody radiation curve that describes the blackbody radiation in the cavity when the cylinder is traveling at relative velocity v, and θ is the angle subtending the absorbed or emitted radiation and the velocity vector $\vec{v}$. The Dopplerization of the radiation within the cylinder will also result in a change in the energy of each photon:

$$h\nu_v = h\nu_o \left[\frac{1 - \frac{v \cos\theta}{c}}{\sqrt{1 - \frac{(v\cos\theta)^2}{c^2}}} \right] \tag{5}$$

When the cylinder is at rest, the energy of a photon (E_o) with peak frequency (ν_o) striking normal to either end wall is given by:

$$E_o = h\nu_o \tag{6}$$

However, when the cylinder is moving at a uniform velocity over a given time, the energy of a photon at the peak of the blackbody radiation curve that is propagating normal to the end walls will *decrease* as a result of being absorbed and emitted by the front end wall ($\theta = 0$) and *increase* as a result of being absorbed and emitted by the back end wall ($\theta = \pi$). As a result of the second order effects, the energy *increase* at the back wall will be greater than the energy *decrease* at the front wall. Consequently, as a result of the uniform translational movement, the energy ($h\nu_v$) of the photons absorbed and emitted by both walls will increase over time dt. As a result of the two absorptions and the two emissions that take place at the front and back walls during dt, the velocity-dependent energy of the photon at the peak of the black body radiation curve is given by:

$$E_{(normal)v} = 2h\nu_o \left[\frac{1 - \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \right] + 2h\nu_o \left[\frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \right] = 2h\nu_o \left[\frac{2}{\sqrt{1 - \frac{v^2}{c^2}}} \right] = h\nu_o \left[\frac{4}{\sqrt{1 - \frac{v^2}{c^2}}} \right] \tag{7}$$

Since the photons in the blackbody radiation can strike the end walls at any angle (θ) from 0 to $\pm\frac{\pi}{2}$ and any angle (φ) from 0 to $\pm\frac{\pi}{2}$, the energy transfer depends on the cosines of the angles of incidence and the angles of emission relative to the normal:

$$E_v = E_{(normal)v} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\varphi = \frac{E_{(normal)v}}{4} \tag{8}$$

Thus the average energy transfer per photon at the peak of the blackbody radiation curve at any angle is one-fourth of the energy transfer of a photon at the peak of the blackbody radiation curve that strikes the end wall perpendicular to the surface. Consequently, when the cylinder is in uniform motion, the average photon with a frequency at the peak of the blackbody radiation curve will have its energy increased in time dt by:

$$E_v = h\nu_v = h\nu_o \left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right] \quad (9)$$

As a result of the Doppler shift expanded to second order, after a given time interval, the energy of the photons interacting with the matte black end walls of the cylinder moving at constant velocity will contribute to a blackbody distribution of radiation characterized by a peak with greater frequency and thus a higher temperature. The *first-order* Doppler effect alone cannot account for the nonlinear relationship between applied energy and kinetic energy because the first-order Doppler shifts at the end wall at the front of the cylinder and the end wall at the back of the cylinder cancel each other and the average frequency of the photons in the cavity does not change [9]. However, the expansion of the Doppler effect to *second order* gives the asymmetry necessary to convert motion into internal energy.

As a result of the Doppler shift expanded to second order, as $v \to c$, the energy of the average photon with a frequency at the peak of the blackbody radiation curve approaches infinity. Since this energy increases at the expense of the conversion of the applied potential energy into the kinetic energy of the cylinder, the increased energy of the radiation can be considered to be the equivalent of a velocity-dependent increase in the apparent inertial mass of the radiation. Since the apparent inertial mass of the blackbody radiation increases as a result of the uniform movement of the cylinder through the vacuum, the applied force ($\boldsymbol{F_{applied}}$) needed to maintain a constant velocity ($\boldsymbol{v}$) cannot be constant but must increase over the time interval dt consistent with Eqn. (3). Thus the apparent increase in the inertial mass is a manifestation of the fact that the applied potential energy is transformed not only into the kinetic energy of the radiation-filled cylinder but also into the internal energy of the blackbody radiation or photon gas itself.

A complete conversion of the applied potential energy into the kinetic energy of the cylinder would only happen when the inside of the cylinder is at absolute zero and there would not be any photons within the cavity [10,11]. As absolute zero is unattainable, this is a limiting condition what would not happen in nature.

The temperature (T) of the photon gas within a cylinder moving at constant velocity is related to the peak frequency of the radiation in the cylinder at rest and on the velocity of the cylinder. It is obtained by taking Eqn. (9) into consideration in order to create the relativistic version of Wien's displacement law given below:

$$E_v = h\nu_v = h\nu_o \left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right] = 2.821439\, kT \quad (10)$$

Where, k is Boltzmann's constant. I assume that while the peak frequency and peak energy change with velocity, the spectral distribution, which is described by Planck's blackbody radiation law, remains invariant. The increase in the temperature (dT) of the blackbody radiation that is related to the increase in the energy (dE_v) of the photons at the peak of the blackbody radiation curve resulting from the Doppler shift is obtained by differentiating Wien's displacement law:

$$dE_v = 2.821439k\, dT \quad (11)$$

Planck's blackbody radiation law relates the spectral distribution of energy to the internal energy. The internal energy (U) of the blackbody radiation is related to the temperature of the blackbody radiation by the Stefan-Boltzmann law:

$$U = \frac{8\pi^5 k^4}{15h^3c^3} VT^4 \quad (12)$$

The increase in the internal energy of the blackbody radiation that is due to the velocity-induced increase in the temperature of the blackbody radiation is given by differentiating Eqn. (12):

$$\mathrm{d}U = \frac{32\pi^5 k^4}{15h^3c^3} VT^3 dT \quad (13)$$

Where, $\frac{8\pi^5 k^4}{15h^3c^3}$ is the product of the Stefan-Boltzmann (σ) constant and one-fourth the speed of light ($a = \frac{c\sigma}{4}$ = 7.56 × 10^{-16} J m^{-3} K^{-4}). Thus part of the applied potential energy used in moving the cylinder through a vacuum causes an increase in the internal energy of the blackbody radiation in the cylinder consistent with the conservation of energy. Thus the amount of potential energy needed to move the cylinder at a given constant velocity is greater than the potential energy predicted to move the cylinder if the internal energy of the radiation did not change with movement, and the radiation did not acquire an apparent inertial mass.

In order to formally relate the increase in the potential energy needed to maintain a constant velocity to the apparent inertial mass, we have to consider the linear momentum of the photons that make up the blackbody radiation. The linear momentum (p) of a photon is given by [12]:

$$p = \frac{E}{c} = \frac{h\nu}{c} \tag{14}$$

When the cylinder is at rest, according to Eqn. (8), the linear momentum transferred by the average photon with peak frequency (ν_o) moving in any direction when they are absorbed or emitted by the end wall is given by:

$$p_o = \frac{h\nu_o}{4c} \tag{15}$$

When the cylinder is in uniform motion, the average photon with the peak frequency (ν_v) traveling in any direction that is absorbed and emitted by the front end wall and then absorbed and emitted by the back end wall, will have its linear momentum increased by:

$$p_v = 2\frac{h\nu_o}{4c}\left[\frac{1-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\right] + 2\frac{h\nu_o}{4c}\left[\frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\right] = 2\frac{h\nu_o}{4c}\left[\frac{2}{\sqrt{1-\frac{v^2}{c^2}}}\right] = \frac{h\nu_o}{c}\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right] \tag{16}$$

After a given duration of time dt, the linear momentum of the photons traveling in any direction in the cylinder moving at uniform velocity would increase. By definition, the linear momentum of the photon with the new peak frequency is equal to the product of its mass (m_v) and velocity:

$$p_v = \frac{h\nu_0}{c}\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right] = m_v v \tag{17}$$

Since the velocity of the photons in the cavity is constant and equal to c, the total differential of the linear momentum is given by:

$$dp_v = c\,dm_v \tag{18}$$

As a result of the Doppler effect expanded to second order, the radiation produces an increase in linear momentum in a given duration of time, on the cylinder moving at constant velocity. The increase in linear momentum results in an increase in the apparent inertial mass (dm) of the photon with peak frequency moving at velocity c in any direction within the cylinder according to the following formula:

$$dp_v = \frac{h}{c}\nu_o\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right] - \frac{h}{c}\nu_o = c\,dm_v \tag{19}$$

The blackbody radiation within the cylinder in uniform translational motion provides a linear momentum that increases with velocity and time. Consequently, the applied force needed to maintain a constant velocity during that duration of time cannot remain constant but must also increase to compensate for the increased linear momentum. Interestingly, while he was working on the quantum nature of radiation, Einstein [12,13] realized that the momentum of radiation would exert a "*radiation friction*" on a moving body, but was too engaged in the General Theory of Relativity to re-interpret the electrodynamics of moving bodies in terms of dynamics.

The apparent increase in the inertial mass of a photon is obtained by dividing all terms in Eqn. (19) by c:

$$\frac{dp_v}{c} = \frac{h}{c^2}\nu_o\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right] = dm_v \tag{20}$$

After substitution of p_v with $\frac{E_v}{c}$ according to [12], we get:

$$\frac{dE_v}{c^2} = \frac{h}{c^2}\nu_o\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right] = dm_v \tag{21}$$

After rearrangement, we get:

$$dE_v = h\nu_o\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right] = c^2 dm_v \tag{22}$$

which is reminiscent of the equation given by Einstein [1].

The increase in the apparent inertial masses of the photons (dm_v) at the peak of the spectral distribution given by Planck's radiation law is related to the total increase in the apparent inertial mass (dm) of the blackbody radiation in the cylinder. The time needed for the photons within the cylinder to go from an initial state whose blackbody radiation is described by a peak at one frequency to a final state whose blackbody radiation is described by a peak at a higher frequency may depend on the details of the force and the dimensions of the cylinder. To simplify matters, I assume that the duration of time dt needed for the blackbody radiation to reach each state is a constant and the force is not constant.

Eqn. (22) describes how a change in the uniform velocity of a radiation-filled cylinder is transformed into a change in the energy and inertia of the radiation inside the cylinder. The change in the energy and inertia of radiation results from the Doppler effect expanded to second order and the relationship between energy and inertia depends on the defined relationship between the energy and momentum of radiation $E_v = p_v c$. Below I will use Planck's blackbody radiation law, a law that I contend holds in any inertial frame, to relate the spectral distribution of radiation at any velocity to the internal energy of the radiation.

In a cylinder containing a photon gas at constant volume, the applied potential energy needed to move the cylinder at constant velocity from state 1 to state 2 is distributed between the increase in the kinetic energy of the cylinder itself and the energy equivalent of the apparent inertial masses of all the photons in the photon gas. Based on Eqn. (22), I consider the increase in the energy equivalent of the apparent inertial masses to be equal to the increase in the internal energy (dU) of the photon gas:

$$c^2 dm = dU \qquad (23)$$

Since the internal energy of a photon gas is related thermodynamically to both thermal energy and pressure-volume energy [14,15], we can try to characterize further the increase in the energy of radiation that occurs when a closed thermodynamic system is moved at constant velocity. In a closed thermodynamic system with constant volume moving at constant velocity, the velocity-dependent increase in internal energy (dU) of the photon gas can be considered thermodynamically as a velocity-dependent increase in thermal energy (dQ) and pressure-volume energy (PdV):

$$dU = dQ - PdV \qquad (24)$$

However, PdV vanishes since the volume is constant by definition. The complete differential of the thermal energy (dQ) is given by the following equation:

$$dQ = TdS + SdT \qquad (25)$$

Where, T is the absolute temperature and S is the entropy of radiation in the cavity with a constant volume (V). Note that the increase in thermal energy in the cylinder with adiabatic walls does not result from heat transfer from a thermal reservoir, as it does in traditional thermodynamic systems [16], but from the dynamic effects of the velocity-induced Doppler effect expanded to second order. The velocity-induced Doppler effect can result in an increase in temperature and/or an increase of entropy. The increase in the thermal energy of the photon gas that happens in the absence of heat flow may help in the kinematic understanding of the contentious and contradictory field of relativistic thermodynamics [17].

If the walls of the cavity are matte black, consistent with Planck's blackbody radiation law, the photon number is not conserved and the increase in internal energy due to the uniform translational motion for a given duration will result in an increase in the temperature (due to an increase in the frequency of the photons at the peak of the blackbody radiation curve) and an increase in entropy (due to an increase in the number of photons). Neither of these quantities alone are considered here to be invariants.

The ability of the blackbody radiation to take up thermal energy is given by the specific heat capacity at constant volume (C_V) that is obtained by differentiating the internal energy at constant volume with respect to temperature:

$$C_V = \left(\frac{dU}{dT}\right)_V = \frac{32\pi^5 k^4}{15 h^3 c^3} \mathrm{V} T^3 \qquad (26)$$

The spectral number density (n) of photons, each with a given frequency, in the cavity at a given temperature (T), is obtained by dividing the internal energy (U), given by Planck's blackbody radiation law, by $h\nu$. Planck's [18] blackbody radiation law gives the spectral energy density:

$$u(\nu,T)d\nu = \frac{U}{V}(\nu,T)d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}}-1}d\nu \quad (27)$$

The spectral number density or the number of photons with a given frequency is given by:

$$n(\nu,T)d\nu = \frac{u(\nu,T)}{h\nu}d\nu = \frac{8\pi h^3\nu^2}{h^3c^3e^{\frac{h\nu}{kT}}-1}d\nu \quad (28)$$

The total number density $(\frac{N}{V})$ of photons within the cylinder varies with the velocity of the cylinder in the same way that the total number density varies with temperature. The total number density of photons comprising a blackbody distribution of radiation characterized by a given temperature at constant volume is obtained by integrating Eqn. (28):

$$\frac{N}{V} = \int_0^\infty n(\nu,T)d\nu = \frac{8\pi k^3T^3}{h^3c^3}\int_0^\infty \frac{h^3\nu^2}{k^3T^3e^{\frac{h\nu}{kT}}-1}d\nu = \frac{8\pi k^3T^3}{h^3c^3}\int_0^\infty \frac{\left(\frac{h\nu}{kT}\right)^2\frac{hd\nu}{kT}}{e^{\frac{h\nu}{kT}}-1} \quad (29)$$

The integral in Eqn. (29) can be solved by letting $x = \frac{h\nu}{kT}$ and $dx = \frac{h}{kT}d\nu$:

$$\frac{N}{V} = \int_0^\infty n(\nu,T)d\nu = \frac{8\pi k^3T^3}{h^3c^3}\int_0^\infty \frac{(x)^2dx}{e^x-1} \quad (30)$$

Since $\int_0^\infty \frac{(x)^2dx}{e^x-1} = 2.404$, the total number density of photons in a cavity of temperature T is:

$$\frac{N}{V} = 60.42\left(\frac{kT}{hc}\right)^3 = 60.42\left(\frac{k}{hc}\right)^3 T^3 = 2.04\times 10^7\, T^3 \quad (31)$$

Differentiating Eqn. (31), we get the change in the total number density of photons with a change of temperature at constant volume:

$$\frac{1}{V}dN = 181.26\left(\frac{k}{hc}\right)^3 T^2dT \quad (32)$$

The total entropy of photons within the cylinder varies with the velocity of the cylinder in the same way that the total entropy varies with temperature. The increase of entropy of the photon gas with temperature T is obtained by dividing the internal energy differential given in Eqn. (13) by the temperature:

$$dS = \frac{dU}{T} = \frac{32\pi^5k^4}{15h^3c^3}VT^2dT \quad (33)$$

Since S vanishes at absolute zero where T also vanishes, the absolute entropy S [19] of the photon gas at any temperature T is obtained by integrating Eqn. (33):

$$S = \frac{32\pi^5k^4}{45h^3c^3}\,VT^3 \quad (34)$$

The entropy density of photons within the cylinder varies with the velocity of the cylinder in the same way that the entropy density varies with temperature. The value of the entropy obtained from Eqn. (34) can be used to find the entropy density of the blackbody radiation in the cylinder characterized by temperature T:

$$\frac{S}{V} = \frac{32\pi^5k}{45}\left(\frac{kT}{hc}\right)^3 \quad (35)$$

By taking the ratio of Eqn. (35) and Eqn. (31), the entropy per photon in the blackbody radiation is found to be a constant independent of velocity and temperature, and is given by:

$$\frac{S}{N} = \frac{SV}{NV} = \frac{32\pi^5k}{45}\left(\frac{kT}{hc}\right)^3\frac{1}{60.42\left(\frac{kT}{hc}\right)^3} = \frac{32\pi^5k}{45}\frac{1}{60.42} = 3.60k \quad (36)$$

How the entropy of a photon relates to the degrees of freedom of a photon is a mystery. If entropy is defined by the degrees of freedom, with each degree of freedom contributing to the entropy, then the entropy of a photon is close to that of a diatomic gas molecule with translational, rotational and vibrational degrees of freedom [20]. For a diatomic gas molecule, the three translational degrees of freedom contribute $\frac{3}{2}k$, the two rotational degrees of freedom contribute $\frac{2}{2}k$ and the two vibrational degrees of freedom contribute $\frac{2}{2}k$ for a total of $3.5k$. If a photon were a point particle with three degrees of freedom for translation and one degree of freedom for polarization, the entropy would be close to $2.5k$. Quantum mechanically speaking [21], entropy is related to the number of microstates (Ω), which for the photon would be 36.6, since $S = k\ln\Omega$. The entropy of a photon given in Eqn. (35) is more consistent with the model of an extended and dynamic photon consisting of translational and rotational oscillators [22-25], and less consistent with the model of a kinematic photon being a mathematical point with integer spin.

The average internal energy of the photons ($\bar{E}$) composing the photon gas in the cylinder is related to the velocity of the cylinder in the same way that the average internal energy of the photons is related to the temperature. The average internal energy of the photons in the blackbody radiation is given by the ratio of the internal energy density given in Eqn. (12) and the total number of photons per unit volume given in Eqn. (31):

$$\bar{E} = \frac{UV}{NV} = \frac{8\pi^5}{15}\frac{1}{60.42}kT = 2.70\ kT \qquad (37)$$

Consistent with the shape of the curve that describes blackbody radiation, the value for the average internal energy of the photon is slightly lower than the value of the internal energy of the photon at the peak of the blackbody radiation curve given by the Wien displacement law:

$$E_{(peak)} = 2.82\ kT \qquad (38)$$

When the walls of the cavity are matte black, both the energy of the photon at the peak of the curve that characterizes the blackbody radiation and the number of photons at the peak of the curve increases as a result of the velocity-induced Doppler effect (Fig. 2). This is best characterized by the velocity-dependent temperature of the photon gas inside the cylinder. The velocity-induced temperature increase has a greater effect on photon number than on the energy of the photon at the peak of the blackbody radiation curve since the energy of the photon at the peak of the blackbody radiation curve is directly proportional to temperature while the number of photons in the blackbody radiation is proportional to the third power of temperature. The velocity-induced Doppler effect expanded to the second order results in an increase in the internal energy density, the temperature, the entropy density, and the apparent inertial mass (Table 1).

When the walls of the cavity are not black, but are perfectly specular reflecting, uniform translational motion will cause an increase in the energy of the photon at the peak of the curve that characterizes the radiation but there will be neither a change in photon number ($dN = 0$) nor entropy ($dS = 0$). Consequently, the radiation in the cavity will no longer be distributed according to Planck's blackbody radiation law and the analysis given here will not hold. Nevertheless, whether the walls of the cavity are matte black or perfectly specular reflecting, the velocity-induced change in the peak of radiation would be irreversible and the potential energy irretrievable because moving the blackbody radiation-filled cylinder at uniform velocity in the opposite direction would not cause a decrease in internal energy but a further increase. Indeed I have shown that the interaction of Doppler-shifted photons with moving bodies is the fundamental and inevitable cause of irreversibility [10].

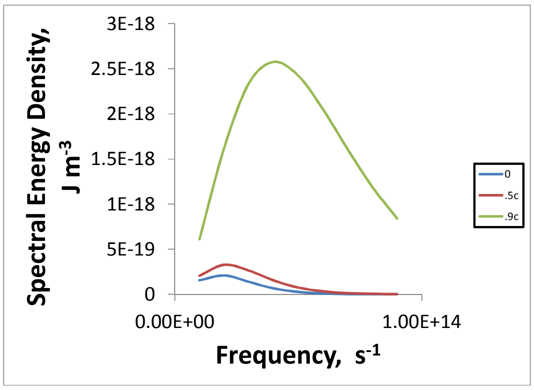

Fig.2: The velocity-induced change in the distribution of blackbody radiation in a cylinder with matte black and adiabatic walls and constant and invariant volume translating uniformly at various speeds (0, 0.5c, and 0.9c). Note that the spectral energy density (in J/m^3) at a given velocity multiplied by the square of the speed of light gives the spectral mass density (in kg/m^3) at a given velocity. Likewise the integral of the spectral energy density (in J/m^3) at a given velocity multiplied by the square of the speed of light gives the total mass density (in kg/m^3) of the enclosed radiation at a given velocity.

As a result of the increase in the internal energy (dU) of the photon gas in the moving cylinder described and explained by the velocity-induced Doppler effect expanded to second order, the blackbody radiation within the cavity can be considered to have a velocity-dependent apparent inertial mass (dm) that provides a resistance to movement at constant velocity (Table 1).

For the cylinder to move at constant velocity, an additional amount of potential energy must be applied to the moving cylinder in order to compensate for the apparent inertial mass of the blackbody radiation within. In light of what we have deduced about the velocity-induced increase in internal energy that will take place as a result of the Doppler effect expanded to second order, we can write the relationship between the applied potential energy and the change in kinetic energy of the radiation filled cylinder like so:

$$PE_{applied} = \frac{1}{2}m\ dv^2 + dU \qquad (39)$$

Where, dU is the differential in the internal energy density that results from the differential velocity change dv. Given that Eqn. (23) states that $c^2 dm = dU$,

$$PE_{applied} = \frac{1}{2} m\, dv^2 + c^2 dm \qquad (40)$$

Table 1: The velocity-induced increase in the frequency (ν), energy ($h\nu$) and momentum ($\frac{h\nu}{c}$) of a photon at the peak of the curve characterizing blackbody radiation; the velocity-induced increase in the temperature (T), internal energy density ($\frac{U}{V}$), photon number density ($\frac{N}{V}$) and entropy density ($\frac{S}{V}$) of the blackbody radiation; and the velocity-induced increase in the apparent mass ($dm = \frac{dU}{c^2}$) of blackbody radiation initially at rest at 300 K for a cylinder with matte black and adiabatic walls and with constant and invariant volume. The values are given for an initial state of zero velocity and a final state with a constant velocity of 0.9c.

v	ν s^{-1}	$E = h\nu$ J	$p = \frac{h\nu}{c}$ kg m s^{-1}	T K	$\frac{U}{V}$ J m^{-3}	$\frac{N}{V}$ m^{-3}	$\frac{S}{V}$ J K^{-1} m^{-3}	dm kg
0	1.76×10^{13}	1.17×10^{-20}	3.90×10^{-29}	300	6.11×10^{-6}	5.45×10^{14}	2.73×10^{-8}	0
0.9c	4.04×10^{13}	2.68×10^{-20}	8.92×10^{-29}	688	168×10^{-6}	65.8×10^{14}	32.9×10^{-8}	1.80×10^{-21}

This means that the average force calculated from the work-energy theorem given in Eqn. (3) to be sufficient to accelerate the cavity with constant mass m at rest to a given velocity will actually be an underestimate of the required force. The additional force necessary is a result of the velocity-dependent increase in apparent inertia (dm) of the photon gas within the cylinder that must also be considered when calculating the relationship between the average applied force ($\boldsymbol{F_{applied}}$) and acceleration ($\boldsymbol{a}$):

$$\boldsymbol{F_{applied}} = (m + dm)\boldsymbol{a} \qquad (41)$$

The optomechanical counterforce ($\boldsymbol{F_{Doppler}}$) resulting from the Doppler effect expanded to second order was already found to describe and explain the nonlinear relationship between applied force and the acceleration of particles with a charge and/or a magnetic moment [11,26]:

$$\boldsymbol{F_{applied}} + \boldsymbol{F_{Doppler}} = m\boldsymbol{a} \qquad (42)$$

and here I have shown that the Doppler effect can describe and explain the nonlinear relationship between applied potential energy and constant velocity for an extended object:

$$E_{applied} + \int_{t_1}^{t_2} \boldsymbol{F_{Doppler}} \boldsymbol{v}\, dt = \int_{t_1}^{t_2} \boldsymbol{F_{applied}} \boldsymbol{v}\, dt \qquad (43)$$

The discrepancy between Hasenöhrl's and Einstein's equations for the relationship between the energy and inertia of radiation has been difficult to reconcile [27-35]. Here I have obtained Einstein's relationship for the energy equivalent of mass based on Hasenöhrl's *gedankenexperiment* by taking into consideration the relationship between the energy and linear momentum of a photon and the dynamic processes that result from the Doppler effect expanded to the second order in Euclidean space and Newtonian time.

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Received: 16 September, 2014
Accepted: 4 March, 2015